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Fuzzy cosets and their gravity duals

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Abstract: Dp-branes placed in a certain external RR $(p+4)$-form field expand into a transverse fuzzy two-sphere, as shown by Myers. We find that by changing the $(p+4)$-form background other fuzzy cosets can be obtained. Three new examples, $S^2 \times S^2$, $\mathbb{CP}^2$ and $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ are constructed. The first two are four dimensional while the last is six dimensional. The dipole and quadrupole moments which result in these configurations are discussed. Finally, the gravity backgrounds dual to these vacua are examined in a leading order approximation. These are multi-centered solutions containing $(p+4)$- or $(p+6)$-dimensional brane singularities.

Keywords: D-branes, Black Holes in String Theory, AdS/CFT Correspondance, Non-Commutative Geometry.

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1. Introduction

The ideas of non-commutative geometry are finding an increasingly prominent role in string theory. For example, as has been known for some time now [1], the coordinates transverse to N D-branes do not commute in general and their dynamics is governed by a non-abelian gauge theory. More recently, it has been found that turning on a world volume U(1) gauge field parallel to the D-branes gives rise to a non-commutative version of Yang-Mills theory.

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In [2, 3], the T-duality properties of D-brane actions were studied. It was shown that on account of the non-abelian nature, the world volume theory for N Dp-branes must couple to Ramond-Ramond (RR) field strengths of degree $p+4$ and above, besides having the well understood couplings to RR fields of degree $\leq p+2$. The extra couplings have interesting consequences. Myers showed that for an appropriate $(p+4)$-form background, the transverse coordinates do not commute in the ground state and the resulting configuration is described by a non-commutative generalization of the of the two-sphere, called the fuzzy sphere. Dispersing the branes in this manner also results in a dipole moment for the $(p+4)$-form field strength. The effect is somewhat analogous to the polarization of a neutral atom when placed in an external electric field: the positive and negative charges of the atom separate in the external field giving rise to a dipole moment.

The Myers effect was investigated in the AdS/CFT context by Polchinski and Strassler [4]. They considered D3-branes placed in the corresponding transverse seven-form field strength background and constructed the supergravity solution dual to the fuzzy sphere. It was found that the solution contains a five brane singularity. In fact the resulting spacetime can be essentially divided into two regions. One, towards the boundary is approximately the multi-centered D3-brane geometry, while the other deep in the interior, corresponding to the infrared in the gauge theory, is the five-brane geometry. The interpolating metric between these two regions is the gravity background dual to the five brane with a world volume U(1) field turned on. This establishes, at least for large 't Hooft coupling, that the infrared (IR) dynamics of the $(3+1)$-dimensional fuzzy sphere vacuum is governed by the $(5+1)$-dimensional five-brane theory with a world volume U(1) field strength. The two kinds of noncommutativity mentioned at the outset above, are therefore related under renormalization group flow, in this system.

The fact that in [3], the transverse coordinates do not commute even in the ground state, brings the noncommutativity of the geometry seen by D-branes into sharp focus. The purpose of this paper is to study situations in which this happens in more generality. The particular background $(p+4)$-form field strength in [3] is proportional to the structure constants of SU(2), (and hence preserves an SO(3) subgroup of the R-symmetry group) and the resulting configuration is a non-commutative generalization of the coset SU(2)/ U(1). We show in this paper that more generally, a background which preserves a subgroup G of the R-symmetry group (in the sense that the $(p+4)$-form field strength is proportional to the structure constants of the group G), gives rise to ground states which are non-commutative generalizations of particular cosets of G. We discuss which cosets can be realized in this manner and show how coherent state techniques are useful for analyzing the fuzzy cosets.

In applying these general considerations to string theory we are faced with a serious limitation: there are too few dimensions to play with! The dimension of G can at most be the number of transverse dimensions, which in turn can be no bigger

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than nine. This allows only three possibilities, SU(2), SU(2) $\times$ SU(2), and SU(3). The first, as Myers showed, gives the fuzzy two-sphere. The second, yields one new surface: fuzzy $S^2 \times S^2$, which is four dimensional. The third, gives rise to two cosets of SU(3), namely SU(3)/ U(2) (also known as $\mathbb{CP}^2$), and $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$. These are four- and six-dimensional manifolds, respectively. The resulting configurations acquire a dipole moment with respect to the $F^{(p+4)}$ field strength. In addition they acquire multipole moments with respect to RR field strengths of higher degree as well. For $S^2 \times S^2$, a quadrupole moment with respect to $(p+6)$-dimensional field strength; for $\mathbb{CP}^2$, a dipole moment with respect to $(p+6)$-dimensional field strength; and finally, for $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$, a dipole moment with respect to the $(p+6)$- and the $(p+8)$-form field strengths. As the number of Dp-branes goes to infinity, the fuzzy surface becomes an increasingly better approximation to the corresponding manifold, and one can think of the Dp-branes expanding into a higher-dimensional brane which wraps this manifold. In our discussion, the coset manifold is always embedded in flat space. As a result no net charge is acquired with respect to the higher-dimensional brane. From the perspective of the higher-dimensional brane, some of the dipole moments as well as Dp-brane charge arises because a topologically non-trivial gauge field is turned on in the world volume theory.

There is one big difference between the S^2 case discussed in [3] and the $S^2 \times S^2$ and the two cosets of SU(3) discussed in this paper. In the former case, as is discussed in [4] supersymmetry can be preserved after adding additional mass terms to the theory. In contrast, in the examples considered here, even allowing for mass terms, supersymmetry is completely broken. The analysis mentioned in the previous paragraph is carried out in the tree-level approximation which is good at weak coupling. What happens at large 't Hooft coupling is much less certain.

To study this question we turn to the dual supergravity description in section 5. Our discussion follows [4] closely and we adopt the same strategy of choosing parameters which allow spacetime to be divided into distinct regions, each governed approximately by one type of D-brane. We find that the gravity backgrounds dual to the fuzzy cosets mentioned above contain singularities which can be interpreted as $(p+4)$- or $(p+6)$-dimensional branes. The presence of singularities is in accord with expectations based on no-hair theorems since perturbing the $\mathcal{N} = 4$ theory is dual to adding hair in the near extremal p-brane geometry. The singularities and their dual descriptions are related to those discussed in [5]–[10].

Our analysis of the gravity solutions is incomplete in one important aspect. Take as an example a region of spacetime governed by the Dp-brane metric which crosses over to the D$(p+2)$-brane geometry. In the solution we construct, we establish that the metrics in different regions agree in the overlap, but only to leading order in $(F^{(p+4)})^2/(F^{(p+2)})^2$ — the ratio of two RR field strengths. This is not enough,

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especially in the absence of supersymmetry. One needs to go to second order at least, before establishing the existence of the solutions. Unfortunately, the analysis gets rapidly complicated and we cannot push it this far. Thus, our discussion of the gravity solutions should be viewed as only the first step in a more definitive study.

There are many further directions to pursue. It would be revealing to understand, by an analysis in the $(p+1)$-dimensional gauge theory at weak 't Hooft coupling, the infrared dynamics in the fuzzy vacuum, in particular if it is governed by a $(p+3)$- (or higher) dimensional theory. Other perturbations of the $\mathcal{N}=4$ theory, especially those which preserve supersymmetry and can therefore be controlled better, are also interesting. Cosets are among the best understood fuzzy surfaces. However, more general perturbations to the gauge theory should yield other kinds of non-commutative surfaces as well. There are close connections between the developments discussed here and those in [11, 12] which should be pursued in more depth. Finally, extending this analysis to non-compact groups G might yield examples of cosmological interest.

One final point. The reader might wonder why we have not considered other variants of the dielectric effect obtained by turning on a $(p+6)$- or higher form field strength. In section 4.4 we briefly discuss one such example which gives rise to a fuzzy generalization of S^4. For the most part though, we postpone a discussion of these cases for future. This is because, such perturbations of the $\mathcal{N}=4$ theory typically result in an unstable theory with runaway directions in field space along which the energy goes to minus infinity. For the trilinear terms considered in this paper, runaway behavior is prevented by the quartic terms present in the $\mathcal{N}=4$ theory. But for the higher form field strengths which couple to operators involving more than four scalars, such instabilities are typically present. In fact, the fuzzy S^4 case mentioned above is an example of this. It is an extremum of the action but not a minimum: along a direction in field space the energy goes to minus infinity. We hope to return to the higher form field strength case in the future. The runaway behavior could well be absent in the full Born-Infeld action, or for some specific choices of RR field strengths and other couplings, which preserve supersymmetry. The instabilities might also be interesting in their own right and could signify higher-dimensional branes decaying to lower-dimensional ones.

Let us end this section by summarizing some additional references. Two good introductions to some of the ideas in non-commutative geometry are [13, 14]. Cosets are discussed from the point of view of coherent states in [15] and from the point of co-adjoint orbits by [16]. The mass deformed $\mathcal{N}=4$ theory was studied in [17]. The fuzzy two-sphere was studied in matrix theory in [18]. Field theories on fuzzy $\mathbb{CP}^2$ have been studied in [19]. There is a dauntingly large literature on AdS/CFT now, starting with [20, 21, 22], much of it is well summarized in [23]. The dynamics of D-branes with a world volume U(1) gauge field and its relation to non-commutative

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Yang-Mills theory was studied in [24, 25]. The gravity duals were discussed in [26, 27, 28]. One recent example of related gravity solutions is, [29]. Other gauge theory deformations of interest have been looked at in [30].

2. Fuzzy surfaces

We start with a brief discussion of fuzzy surfaces and some related ideas in non-commutative geometry. A readable account of these topics can be found in [13, 14]. Here we will settle for a brief pedestrian account of the subject.

The essential idea behind fuzzy surfaces is that the position coordinates of the manifold are no longer commuting variables but instead became operators satisfying an algebra. For example, the coordinates on classical phase spaces can be thought of as operators analogous to physical observables in quantum mechanics, and the algebra they satisfy as being the analog of the algebra of quantum mechanical observables.

More precisely, a fuzzy surface (see [13]) may be defined as a sequence of algebras A_N which form an increasingly better approximation to the algebra of continuous functions on some manifold X. Concretely one can think of the algebra A_N in terms of matrices M_N. The eigenvalues of these matrices (and more generally expectation values of products of matrices) can be compared with corresponding quantities in the continuous manifold. The $N \to \infty$ limit is a classical limit where the expectation values of the matrices M_N agree arbitrarily well with the corresponding quantities in the classical manifold.

As an example we consider first the fuzzy S^2 surface. The manifold S^2 can be defined by embedding it in R^3 by the relation:

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = 1 \,. \tag{2.1}$$

Let J^1, J^2, J^3 be the three angular momentum operators in the spin j representation of SU(2). These satisfy the relation:

$$J^i J^i = (J^1)^2 + (J^2)^2 + (J^3)^3 = j(j+1) \,. \tag{2.2}$$

One can think of the matrices $\hat{X}^i \equiv J^i/\sqrt{j(j+1)}$ as non-commutative generalizations of the coordinates X^i; the relation (2.2) can then be identified with (2.1). It is easy to see that the limit $j \to \infty$ is a classical limit: the expectation values of any product of the matrices $\hat{X}^i$ agrees with the corresponding quantity in S^2 upto corrections of order $1/j$.

The sphere is also a coset SU(2)/ U(1). By generalising the discussion for the sphere, one can construct non-commutative analogues for some, though not all, coset manifolds. Let us explain which fuzzy cosets can be obtained in this manner.[1] For

[1] Although many of these ideas are more general, we will restrict ouselves to compact groups in this paper.

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any compact group G consider a representation R and a weight vector $|\mu\rangle$ in this representation. The isotropy group $H_{|\mu\rangle}$ of $|\mu\rangle$ is defined to be the subgroup of G which leaves $|\mu\rangle$ invariant upto a phase. Now for any G consider the isotropy group $H_{|\mathrm{lws}\rangle}$ of the lowest weight state[2] in some irreducible representation of G. Then the coset $G/H_{|\mathrm{lws}\rangle}$, for any irreducible representation, can be realised as a fuzzy surface. We give a brief argument showing this below. Before proceeding let us make three comments. First, note that for SU(2) the isotropy group for the lowest weight vector in any irreducible representation is U(1). In general though, depending on the representation chosen $H_{|\mathrm{lws}\rangle}$ can be different. Second, it is obvious that $H_{|\mathrm{lws}\rangle}$ must always contain the maximal torus of the group. Third, one can show that the cosets, $G/H_{|\mathrm{lws}\rangle}$, are all even-dimensional manifolds with a symplectic form. In fact they are complex, homogeneous Kähler manifolds [15].

To obtain a fuzzy generlisation of the coset manifold $G/H_{|\mathrm{lws}\rangle}$ we considers a sequence of irreducible representations, labelled by a parameter N, all of which have lowest weight states with the same isotropy group $H_{|\mathrm{lws}\rangle}$. The dimension of each representation in the sequence increases as N increases and goes to infinity in the limit $N \to \infty$. Let the generators in the representation N be denoted as $T^i_N, i = 1, \cdots \dim\ G$. Then one can show that the limit $N \to \infty$ is a classical limit analogous to the $j \to \infty$ limit in (2.2), and the matrices T^i_N in this limit describe the coset manifold $G/H_{|\mathrm{lws}\rangle}$.

We will briefly sketch an argument which makes this plausible for the case of fuzzy S^2. Since we need to establish that the limit is classical it is useful to think in terms of coherent states. For other coset manifolds, the arguments are similar.

As discussed in the appendix, the SU(2) coherent states are of the form

$$|\xi\rangle = \frac{1}{(1+|\xi|^2)^j} e^{\xi J_+} |j, -j\rangle \,, \tag{2.3}$$

where $|j, -j\rangle$ is the lowest weight vector of the representation of SU(2) labelled by half-integer j, J_+ is the raising operator $J_+ = J^1 + iJ^2$, and ξ a complex number. The resolution of unity may be written as

$$\int d\mu_j(\xi) |\xi\rangle\langle\xi| = \mathbb{1} \,, \quad \text{where} \quad d\mu_j(\xi) = \frac{2j+1}{\pi} \frac{d^2\xi}{(1+|\xi|^2)^2} \,. \tag{2.4}$$

This allows us to expand any state in terms of coherent states. Corresponding to any operator $\mathcal{O}$ in the Hilbert space, we can associate a "classical" function $\mathcal{O}(\xi, \bar{\xi}) = \langle\xi|\mathcal{O}|\xi\rangle$. For example,

$$\begin{aligned} \langle\xi|\hat{X}^3|\xi\rangle &= -\left(\frac{j}{\sqrt{j(j+1)}}\right) \frac{1-|\xi|^2}{1+|\xi|^2} \,, \\ \langle\xi|\hat{X}^+|\xi\rangle &= -\left(\frac{2j}{\sqrt{j(j+1)}}\right) \frac{\bar{\xi}}{1+|\xi|^2} \,, \quad \text{etc.} \end{aligned} \tag{2.5}$$

[2]Equivalently, one could have chosen the highest weight state.

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It is now immediately clear as to why the $\hat{X}^i$'s go over into the coordinates in the limit of large j: we simply get the stereographic projection.

The trace of the operator $\mathcal{O}$ can be calculated in terms of coherent states as

$$\mathrm{Tr}\,\mathcal{O} = \sum_{m=-j}^{j} \langle j,m|\mathcal{O}|j,m\rangle = \int d\mu_j(\xi)\langle\xi|\mathcal{O}|\xi\rangle = \int d\mu_j(\xi)\mathcal{O}(\xi,\bar{\xi})\,. \tag{2.6}$$

3. The dielectric effect

By analyzing the T-duality properties of Dp-brane actions it was argued [2, 3] that RR potentials of degree greater than $p+1$ also couple to the world volume theory. Let us briefly recall the arguments. Take the Dp-brane world volume to be oriented along the $0,1,\ldots,p+1$ directions. The coupling to the $(p+4)$-form field strength, which will be the one of main interest in this paper, then takes the form:

$$V_1 = -i\lambda^{-1}\frac{T_p}{3}\int \mathrm{Tr}(X^iX^jX^k)\; F^{(p+4)}_{01\cdots\,pijk}\; dx^0dx^1\cdots dx^p\,. \tag{3.1}$$

Here X^i are the scalars transverse to the brane world volume and are in the adjoint representation of $\mathrm{SU}(N)$. The tension T_p of the D-brane and λ are

$$T_p = \frac{2\pi}{g_s(2\pi l_s)^{p+1}}\,, \qquad \lambda = 2\pi l_s^2\,. \tag{3.2}$$

In addition the scalar potential for the X^i's has a quartic term required by $\mathcal{N}=4$ supersymmetry. Adding it gives a total potential

$$\begin{aligned} V = {}& -\frac{T_p}{4\lambda^2}\sum_{a,b}\int d^{p+1}x\;\mathrm{Tr}\left([X^a,X^b]^2\right) - \\ & - i\frac{T_p}{3\lambda}\int \mathrm{Tr}(X^iX^jX^k)\; F^{(p+4)}_{01\cdots pijk}\; dx^0dx^1\cdots dx^p\,. \end{aligned} \tag{3.3}$$

The first term in (3.3) is invariant under the $\mathrm{SO}(9-p)$ R-symmetry group of rotations in the $9-p$ transverse directions. If in addition the $(p+4)$-form RR field strength is a constant and of the form

$$F^{(p+4)}_{01\cdots ijk} = \begin{cases} -\frac{2}{\lambda}f\epsilon_{ijk}\,, & \text{for } i,j,k \in \{p+1,p+2,p+3\} \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.4}$$

where f is a real constant, then (3.3) preserves a $\mathrm{SO}(3)\times\mathrm{SO}(6-p)$ subgroup of the full R-symmetry group.

Minimizing (3.3) with respect to X^i gives the equations:

$$\left[[X^i,X^j],X^j\right] + if\epsilon_{ijk}\left[X^j,X^k\right] = 0\,. \tag{3.5}$$

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These can be solved by setting

$$X^{p+i} = f J^i\,, \quad i \in \{p+1, p+2, p+3\}\,, \tag{3.6}$$

where J^i belong to an N-dimensional representation of the SU(2) algebra.[3] Note that in the ground state the scalars X^i do not commute. In fact, as discussed in the previous section, X^i represent a non-commutative generalization of the two-sphere.

The quadratic invariant[4] C_r can be used to define the radius R of the fuzzy two-sphere:

$$X^i X^i \equiv R^2 \mathbb{1} = f^2 C_r \mathbb{1}\,. \tag{3.7}$$

The configuration (3.6) has a dipole moment with respect to the $F^{(p+4)}$ form field strength. From (3.1) we see that the dipole tensor P_{ijk} is given by

$$P_{ijk} = -i\lambda^{-1} T_p \int \mathrm{Tr}\left([X^i, X^j] X^k\right). \tag{3.8}$$

and is not zero for $i, j, k \in \{p+1, p+2, p+3\}$ and all distinct. Thus the externally imposed $(p+4)$-form field strength results in a dipole moment, analogous to the polarization of a neutral atom placed in an electric field.

A few comments are worth making about the solution (3.6). First, (3.6) is different from a multi-centered solution in which the N branes are uniformly distributed over the two-sphere. In (3.6) the X^1, X^2 and X^3 coordinates do not commute and hence a definite location in all the three directions cannot be simultaneously assigned to the Dp-branes. A gauge-invariant way to characterize the difference between the two configurations is the following. In the multi-centered solution, the gauge theory is in the Coulomb phase, while in (3.6) it has a mass gap; the heaviest gauge bosons have a mass $M \sim R/l_s^2$ while the lightest have a mass $R/l_s^2\sqrt{N}$.

Second, the energy in the minimum (3.6) is

$$V_N = -\frac{T_p}{6\lambda^2} f^4 N C_r\,. \tag{3.9}$$

In general, there are several different representations of SU(2) with dimension N. The irreducible representation, with the biggest Casimir invariant, has the largest radius and the lowest energy:

$$E_N = -\frac{T_p f^4}{\lambda^2} \frac{N(N^2-1)}{24}\,. \tag{3.10}$$

A reducible representation of the form

$$X^{p+i} = f J^i_{m\times m} \otimes \mathbb{1}_{n\times n}\,, \quad i = 1, 2, 3 \tag{3.11}$$

[3]The remaining $(6-p)$ scalars must commute with the three X^{p+i}'s in (3.6) and with each other.
[4]The quadratic Casimir invariant C_r in the rth representation of SU(2) is defined by $J^i J^i = C_r \mathbb{1}$.

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has smaller size and higher (but still negative) energy:

$$E_m = -\frac{T_p f^4}{24\lambda^2} N(m^2 - 1) \,. \tag{3.12}$$

In contrast, the trivial representation, in which all the transverse scalars have zero expectation values, has zero energy.

Third, in the large-N limit the fuzzy surface approximates S^2 and the Dp-brane configuration increasingly looks like $(p+2)$-dimensional branes wrapped on the sphere. One might ask how the various charges arise from the higher brane's perspective. The configuration carries no $(p+2)$-brane charge, since the sphere in question can be embedded in $\mathbb{R}^3$.[5] Instead, wrapping on S^2 gives rise to a dipole moment. It turns out that the irreducible representation (3.10) corresponds to a single wrapped $(p+2)$-dimensional brane, while the reducible representation (3.11) corresponds to n wrapped $(p+2)$-dimensional branes. The Dp-brane charge arises due to a world volume U(1) magnetic field with components parallel to the two-sphere. This magnetic field carries magnetic monopole number equal to the number of Dp-branes. As was mentioned in the introduction, [4], studied a supersymmetric version of (3.5) for $p = 3$, at large 't Hooft coupling and large N. They found that the low-energy dynamics of the theory was governed by the 5-brane theory. It would be interesting to establish this for small 't Hooft coupling, by a direct analysis of the gauge theory, in the energy regime, $R/l_s^2 > E > R/(\sqrt{N} l_s^2)$.

Finally, it is worth considering what happens when a mass term of the form,

$$V_m = \frac{m^2}{2} \sum_{i=p+1}^{9} (X^i)^2 \,, \tag{3.13}$$

consistent with the SO(3) $\times$ SO($6-p$) symmetry, is added to (3.3). Putting in the ansatz

$$X^{p+i} = aJ^i \quad \text{for } i = 1, 2, 3 \tag{3.14}$$

gives the total energy to be

$$V = \frac{T_p}{2\lambda^2} C_r N \left(a^4 - \frac{4}{3} f a^3 + m^2 \lambda^2 a^2 \right) . \tag{3.15}$$

The energy is minimized at

$$a = \frac{f + \sqrt{f^2 - 2m^2\lambda^2}}{2} \tag{3.16}$$

and at $a = 0$. For $m^2\lambda^2$ less than (great than) $\frac{4}{9} f^2$ the energy at the minimum (3.16) is negative (positive) and lower (larger) than that at the origin. For large enough

[5] Locally, there is D$(p+2)$-brane charge density, but the contributions from the anti-podal points of the sphere cancel out leaving no net charge.

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mass the discriminant in (3.16) changes sign and the minimum (3.16) disappears. Thus the dielectric effect is stable with respect to adding small enough mass terms in the potential. The case $m^2\lambda^2 = \frac{4}{9}f^2$ is clearly special. In this case the potential can be written as a perfect square and the energy at the minimum, (3.16), is zero and equal for all solutions (3.6). In fact, adding supersymmetry preserving mass terms to the $\mathcal{N}=4$ theory gives rise to this case [17].

4. Generalized dielectric effect

4.1 The general case

The essential features of the above discussion are that one started with an external $(p+4)$-form field strength which preserved an $\mathrm{SO}(3)\times\mathrm{SO}(6-p)$ subgroup of the R-symmetry. This gave rise to a solution which can be interpreted as a non-commutative generalization of the surface $\mathrm{SU}(2)/\,\mathrm{U}(1)$.

We are now ready to generalize this discussion. Start with the $(p+4)$-form:

$$F^{(p+4)}_{01\cdots pabc} = -\frac{2}{\lambda} f f_{abc}\,, \tag{4.1}$$

where f_{abc} are the structure constants of some compact group G. The potential (3.3) is then minimized when X^i are of the form:

$$X^{p+i} = fT^i\,, \quad i=1,\ldots,\dim G\,, \tag{4.2}$$

where T^i denote the generators of the group G in some representation of dimension N (the rest of the transverse scalars are proportional to the identity matrix). The discussion in section 2 shows that one can always associate a fuzzy surface with the solution (4.2).

These fuzzy surfaces are non-commutative generalizations of certain cosets G/H of G. H is determined by the choice of representation. For an irreducible representation, H is the isotropy subgroup of the lowest weight state. Different irreducible representations can correspond to different isotropy subgroups, H, and thus different cosets G/H. As mentioned in section 2, that the cosets obtained in this manner are all Kähler manifolds, and we denote the Kähler form by K.

The surface corresponding to (4.2) has dimension $d=\dim G-\dim H$. In general, the configuration (4.2) carries dipole moment (3.8). In addition when $d>2$, dipole moments for higher degree field strengths $F^{(p+6)},\ldots F^{(p+d+2)}$ are also induced; these are defined analogous to (3.8).

Several features of the discussion for the fuzzy two-sphere carry over in more generality as well. The quadratic Casimir invariant C_r of G can be used to assign a "radius" R to the resulting surface:

$$X^iX^i \equiv R^2\mathbb{1} = f^2C_r\mathbb{1}\,. \tag{4.3}$$

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The vacuum energy can also be expressed in terms of C_r as:

$$E = -\frac{T_p}{12\lambda^2} f^4 N C_A C_r \,, \tag{4.4}$$

where C_A is the Casimir invariant in the adjoint representation of G. Once again we see that the bigger surfaces are also of lower energy. For large N the fuzzy surface becomes a good approximation to the manifold G/H. The various dipole moments which couple to $F^{(p+d+2)}, \ldots F^{(p+4)}$ and the Dp-brane charge which couples to $F^{(p+2)}$ can be understood in terms of a $(p+d)$-brane wrapping G/H. For this purpose it is necessary to excite a U(1) gauge field on the world volume of the $(p+4)$-brane. This field strength is given by

$$\mathcal{F}_{z_i \bar{z}_j} = K_{z_i \bar{z}_j} \,. \tag{4.5}$$

Finally, the solutions (4.2) are stable when small mass terms are added. But there is a critical value for the mass beyond which the fuzzy surface vacua disappear.

In applying these general considerations to D-branes there is one immediate constraint. There can be at most nine directions transverse to a D-brane. For a compact group G this leaves only three possibilities: SU(2), SU(2) $\times$ SU(2) and SU(3). The first choice gives rise to the fuzzy S^2 discussed in the previous section. SU(2) $\times$ SU(2) gives rise to fuzzy $S^2 \times S^2$. SU(3) yields two cosets SU(3)/ U(2) (also known as $\mathbb{CP}^2$) and $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$.

One important remark needs to be made before proceeding further. In the previous section we mentioned that the trilinear terms giving rise to the fuzzy S^2 can be made supersymmetric after adding appropriate mass terms. In contrast one finds that, even allowing for mass terms, the $S^2 \times S^2$ and cosets of SU(3) cases break all supersymmetries. While we will not give any details here, one can verify that even the minimal supersymmetry corresponding to one real supercharge in $0+1$ dimensions is not allowed in these cases. The analysis above used the tree-level potential and is valid at weak coupling. However, in going to the large 't Hooft coupling the absence of supersymmetry becomes a serious limitation. We will examine this region of parameter space in the dual gravity description in section 5.

We now turn to discussing the $S^2 \times S^2$ and cosets of SU(3) in more detail.

4.2 $S^2 \times S^2$

Here, $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and the $(p+4)$-form field strength (4.1) is:

$$F^{(p+4)}_{01\cdots pijk} = \begin{cases} -\frac{2}{\lambda} f_1 \epsilon_{ijk} & \text{if } i,j,k \in \{p+1, p+2, p+3\}\,; \\ -\frac{2}{\lambda} f_2 \epsilon_{ijk} & \text{if } i,j,k \in \{p+4, p+5, p+6\}\,; \\ & 0 \text{ otherwise}\,. \end{cases} \tag{4.6}$$

The solution to (3.3) is:

$$X^{p+i} = f_1 J^i_{m\times m} \otimes \mathbb{1}_{n\times n}\,, \quad i = 1,2,3\,, \tag{4.7}$$

$$X^{p+3+i} = f_2 \mathbb{1}_{m\times m} \otimes J^i_{n\times n}\,, \quad i = 1,2,3 \tag{4.8}$$

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with

$$mn = N\,. \tag{4.9}$$

These represent the fuzzy $S^2 \times S^2$ surface, with radii mf_1 and nf_2, respectively. Notice that for given f_1 and f_2, the vacua form a one parameter family of solutions labelled by the integer m. The energy of these solutions is:

$$E = -\frac{T_p N}{24\lambda^2}\Big(f_1^4(m^2-1) + f_2^4(n^2-1)\Big)\,. \tag{4.10}$$

Depending on the ratio f_1/f_2 the lowest energy state has either $m=1$, or $n=1$ and corresponds to a single fuzzy S^2. When $mf_1 \ll nf_2$ one of the two spheres becomes small and (4.7) and (4.8) approach a single fuzzy two-sphere.[6]

The configuration (4.7) and (4.8) gives rise to a dipole moment for the $(p+4)$-form. This arises just as in the single S^2 case. In addition there is a quadrupole moment for the $(p+6)$-form field strength. The quadrupole moment is determined by a coupling of the form:[7]

$$\frac{T_p}{\lambda^2}\int d^0x\cdots d^px\;\partial_n F^{(p+6)}_{0\cdots pijklm}\,\mathrm{Tr}\Big([X^i,X^j][X^k,X^l]X^mX^n\Big)\,. \tag{4.11}$$

One can check that for (4.7) and (4.8) this coupling does not vanish. From the perspective of a $(p+4)$-brane wrapped on $S^2\times S^2$ the various charges and moments arise as follows. A U(1) field $\mathcal{F}$ is turned on in the world volume of the $(p+4)$-brane. As mentioned in the previous section $\mathcal{F} = K$, the Kähler form for $S^2\times S^2$. The Dp-brane charge is determined by $\int_{S^2\times S^2}\mathcal{F}\wedge\mathcal{F}$, and the D$(p+2)$-brane dipole moment by $\int_{S^2}\mathcal{F}$ on the two S^2's. One finds that the solution (4.7) and (4.8) corresponds to n D$(p+2)$-branes wrapping the first S^2 and m D$(p+2)$-branes wrapping the second S^2. To see how the quadrupole moment arises one can think of wrapping the two S^2's in turn. Wrapping the $(p+4)$-brane on the first S^2 gives rise to a dipole moment for the $F^{(p+6)}$ form. Wrapping further on the second S^2 gets rid of this dipole moment but generates a quadrupole moment instead.

4.3 Cosets of SU(3)

4.3.1 General features

We turn next to the case obtained by taking $G = \mathrm{SU}(3)$. Since we need at least eight transverse coordinates, this can be realized only in the D1- or D0-brane theory. The solution (4.2) corresponds to taking eight of the transverse coordinates to be in

[6]In fact, in general, another solution to (3.3) is obtained by setting one set of three coordinates, say X^{p+i}, $i=1,2,3$ to equal (4.7), while the complimentary set of three coordinates (and other transverse coordinates) are proportional to $\mathbb{1}_{N\times N}$. This configuration corresponds to a single fuzzy two-sphere.

[7]Strictly speaking a trace term should be removed in the definition of the quadrupole moment. Imposing the source free equations for $F^{(p+6)}$ will do this automatically.

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an N-dimensional representation of SU(3). Irreducible representations of SU(3) are parametrized by two integers (n, m) (corresponding to the number of fundamental and anti-fundamental indices). One can show that the lowest weight vector in the completely symmetric representation, $(m, 0)$ or $(0, m)$ has an isotropy group $H = \mathrm{U}(2)$.[8] Thus when X^{p+i} as defined in (4.2) are in the representation $(m, 0)$ or $(0, m)$, one gets the non-commutative version of $\mathbb{CP}^2$. For all other (n, m) one can show that the isotropy group is the maximal torus $\mathbb{T}^2$ of SU(3), resulting in fuzzy $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$. Reducible representations correspond to taking a (disjoint) union of surfaces.

Before proceeding let us clarify one point. Strictly speaking, as was discussed in section 2, a fuzzy surface corresponds to a sequence of representations. Any irreducible representation (n, m) can be regarded as an element of a sequence where n/m is kept fixed and $m \to \infty$. In our discussion above, we have implicitly assumed such a sequence.

We now discuss the two cosets in some more detail. The representation (n, m) has dimension

$$D(n, m) = \frac{(n+1)(m+1)(n+m+2)}{2}\,. \tag{4.12}$$

Choosing the symmetric representation $(0, m)$, and setting the total dimension equal to N gives $m^2 \sim 2N$ for large N. The Casimir of this representation can be calculated to be

$$C_{(0,m)} \simeq \frac{m^2}{3} \simeq \frac{2N}{3}\,. \tag{4.13}$$

This gives, on setting $C_A = 3$ in (4.16) an energy for the $\mathbb{CP}^2$ surface,

$$E_{(0,m)} \simeq -\frac{T_p}{\lambda^2}\frac{f^2}{12}Nm^2 \simeq -\frac{T_p}{\lambda^2}\frac{f^2}{6}N^2\,, \tag{4.14}$$

and a radius

$$R^2 \simeq \frac{2}{3}f^2 N\,, \tag{4.15}$$

for large N.

For the representation (n, m), in the limit of large N with n/m fixed, we get that $n, m \simeq N^{1/3}$. The corresponding Casimir $C_{(n,m)} \sim N^{2/3}$ leading to an energy,

$$E_{(n,m)} \sim -\frac{T_p f^4}{\lambda^2}N^{5/3}\,, \tag{4.16}$$

and a radius

$$R^2 \sim f^2 N^{2/3}\,. \tag{4.17}$$

From (4.14) and (4.16) we see that the $\mathbb{CP}^2$ surface has the lowest energy and the the largest size. The energy is lower by a power of $N^{1/3}$, while the radius is larger by a power of $N^{1/6}$ for large N. Reducible representations all have an energy

[8] It is clear that any vector in the fundamental representation has a U(2) isotropy group. It then follows that the lowest weight vector in the symmetric representation of m anti-fundamentals must also have the same isotropy group.

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which is higher than the symmetric representation $(m, 0)$. For example, the reducible representation containing k copies of the symmetric representation has energy

$$E \simeq -\frac{T_p}{\lambda^2}\frac{f^4}{6}\frac{N^2}{k}\,. \tag{4.18}$$

In appendix B and C, we discuss how the Kähler form for the two manifolds can be calculated from the appropriate representations using coherent state techniques. The metric of $\mathbb{CP}^2$ is the well-known Fubini-Study metric and is a generalization of the round metric for S^2. It has only one free parameter, the overall scale which is fixed by the radius R. The metric for $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times \mathrm{U}(1)}$ depends on two parameters. The coherent state techniques yield the Kähler form in variables where the two parameters directly correspond to the values (n, m) used to specify the representation above. In addition, the representations (4.2) yield an embedding of the two surfaces in 4 and 6 dimensions, respectively. The specific form of this embedding is also presented in appendix A. As mentioned in the appendix, we see that in the limit when $m \to \infty$ and $n/m \to 0$, the $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times \mathrm{U}(1)}$ surface degenerates to $\mathbb{CP}^2$.

4.3.2 Dipole moments

We conclude this section with a discussion of the various dipole moments and charges induced in the two cases. The $\mathbb{CP}^2$ case is presented in some detail first, the $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times \mathrm{U}(1)}$ results which can be obtained in the same way, are discussed more briefly at the end. In the following discussion we set $p = 0$ for simplicity, so that we are dealing with a D0-brane. Nothing essential changes in the D1-brane context. In the $\mathbb{CP}^2$ case one expects the configuration (4.2) to carry a dipole moment for the $F^{(6)}$ and $F^{(4)}$ form field strengths and a D0-brane charge. The two dipole moments are determined by the couplings:

$$\begin{aligned} &\frac{T_p}{\lambda^2}\int dt\, F^{(6)}_{0ijklm}\,\mathrm{Tr}\left([X^i, X^j][X^k, X^l]X^m]\right) \\ \text{and}\qquad & i\frac{T_p}{\lambda}\int dt\, F^{(4)}_{0ijk}\,\mathrm{Tr}\left([X^i, X^j]X^k\right), \end{aligned} \tag{4.19}$$

respectively.[9] From the four-brane perspective, no charge for the $F^{(6)}$ field strength arises because the $\mathbb{CP}^2$ surface is embedded in $\mathbb{R}^8$. The D0-brane charge and dipole moments arise due to wrapping a *single* four-brane on $\mathbb{CP}^2$ (to get k four-branes one

[9] Our conventions are that in the expression for the $F^{(6)}$ brane dipole moment each distinct pair of commutators appears only once, i.e.

$$\begin{aligned} F^{(6)}_{0ijklm}\,\mathrm{Tr}\left([X^i, X^j][X^k, X^l]X^m]\right) = F^{(6)}_{012345}\Big(&\mathrm{Tr}\left([X^1, X^2][X^3, X^4]X^5\right) - \\ &-\mathrm{Tr}\left([X^1, X^3][X^2, X^4]X^5\right) + \mathrm{Tr}\left([X^2, X^3][X^4, X^5]X^1\right) + \cdots\Big)\,. \end{aligned} \tag{4.20}$$

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needs to begin with a reducible representation containing k copies of the symmetric representation). A U(1) field strength is turned on in the world volume of the four-brane. It is

$$\mathcal{F}_{z_i\bar{z}_j} = K_{z_i\bar{z}_j}\,, \tag{4.21}$$

where $K_{z_i\bar{z}_j}$ the Kähler form for $\mathbb{CP}^2$ is given in (B.13) of appendix B. This gauge field has a non-trivial first and second Chern class.[10]

The D0-brane charge in the four-brane theory is given by

$$N = \frac{T_{p+4}}{2T_p}\lambda^2 \int \mathcal{F}\wedge\mathcal{F}\,. \tag{4.22}$$

Evaluating the r.h.s. gives an answer $m^2/2$ (for $m \gg 1$), this agrees with the dimension of the representation, N, from (4.12).

The dipole moments for $F^{(6)}$ and $F^{(4)}$ are given by the couplings:

$$\begin{aligned} & T_{p+4}\int F^{(6)}_{0ijklm} X^m dX^i\wedge dX^j\wedge dX^k\wedge dX^l\; dt \\ \text{and}\qquad & T_{p+4}\,\lambda\int F^{(4)}_{0ijk} dX^i\wedge dX^j\wedge \mathcal{F}\; X^k\; dt\,. \end{aligned} \tag{4.23}$$

The integrals above are understood as being done over the $\mathbb{CP}^2$ manifold. In (B.15) we describe how $\mathbb{CP}^2$ is embedded in $\mathbb{R}^8$. This embedding determines X^m and the differential dX^m as functions of $(z_i,\bar{z}_i)$ and their differentials. Also, $\mathcal{F}$ is determined as a function of z_i from (4.21).

One can show that (4.23) agrees quantitatively (in the classical limit, for large m) with (4.22) for any perturbation $F^{(6)}$ and $F^{(4)}$.[11] To show this it is convenient to use the coherent state basis, described in appendix B to evaluate the trace in (4.19), which can be expressed as:

$$\begin{aligned} & \frac{T_p}{\lambda^2}\frac{m^2}{4\pi^2}\int \frac{d^2z_1 d^2z_2}{(1+|z_1|^2+|z_2|^2)^3} dt\; F^{(6)}_{0ijklm}\langle z_i|[X^i,X^j][X^k,X^l]X^m|z_i\rangle \\ \text{and}\qquad & i\frac{T_p}{\lambda}\frac{m^2}{4\pi^2}\int \frac{d^2z_1 d^2z_2}{(1+|z_1|^2+|z_2|^2)^3} dt\; F^{(4)}_{0ijk}\langle z_i|[X^i,X^j]X^k|z_i\rangle\,, \end{aligned} \tag{4.24}$$

respectively. Now it is straightforward to show that the contribution to (4.23) and (4.24) from each point in $\mathbb{CP}^2$ as parametrized by (z_1,z_2) agree. In fact, since $\mathbb{CP}^2$ is a coset, all points on it can be related by the action of the group SU(3) and it is enough to prove that these contributions agree at some one point in the manifold. The calculation is greatly simplified by choosing this point to $z_1 = z_2 = 0$, since the corresponding coherent state is then the lowest weight state itself. As an example

[10] In fact $\pi_2(\mathrm{SU}(3)/\mathrm{U}(2)) = Z$, and $\int_{S^2}\mathcal{F} = m$.

[11] Here we mean a perturbation about the background (4.1).

consider the dipole moment for $F^{(4)}$. The contribution to (4.24) from the vicinity of this point (in the classical limit) is

$$i\frac{T_p}{\lambda}\frac{m^2}{4\pi^2}d^2z\; d^2\omega F^{(4)}_{0ijk}\;\langle\mathrm{lws}|[X^i,X^j]|\mathrm{lws}\rangle\; X^k(0)\,. \tag{4.25}$$

Using the SU(3) algebra, one can show that for a general $F^{(4)}$ this agrees with the corresponding contribution in (4.23).

Before proceeding, let us make one parenthetical remark which will be of relevance in the supergravity discussion of section 5. From (B.14) and (B.15), we see that at $z_1 = z_2 = 0$ the coordinates X^1, X^2, X^4, X^5 lie along the $\mathbb{CP}^2$ surface. The contribution from the neighborhood of this point to the dipole moments coupling to $F^{(4)}_{012k}$ and $F^{(4)}_{045k}$ are in the ratio $\langle\mathrm{lws}|[X_1,X_2]|\mathrm{lws}\rangle/\langle\mathrm{lws}|[X_4,X_5]|\mathrm{lws}\rangle$. From the SU(3) algebra and (B.14) we see that these are equal. In the supergravity dual a B-field will be turned on in the vicinity of the $(p+4)$-brane. This field has rank four and will be specified by two parameters b_1, b_2. From the argument just given, one can argue that $b_1/b_2 = 1$.

Similarly in the $\frac{\mathrm{SU(3)}}{\mathrm{U(1)\times U(1)}}$ case, we start with D0-branes for simplicity, and the configuration (4.2) gives rise to dipole moment for $F^{(8)}$, $F^{(6)}$ and $F^{(4)}$. These can be calculated both from the perspective of the D0-brane theory, in the form of traces over matrices as in (4.19), (4.24), and in the six-brane theory by couplings analogous to (4.23) (once again a U(1) gauge field $\mathcal{F}$ given by the Kähler form (C.7), is turned on in the world volume theory of the six-brane). The resulting answers agree quantitatively. In the supergravity dual the geometry contains a $(p+6)$-dimensional brane. In the vicinity of this brane a B-field is turned on which is characterized by three parameters b_1, b_2 and b_3. As in the $\mathbb{CP}^2$ case, their ratios can be calculated by considering the corresponding local contributions to the dipole moments in the gauge theory. These are determined by the integers (n,m) which characterize the representation.

4.4 Fuzzy S^4

So far we have considered only $F^{(p+4)}$-form field strength backgrounds. We end this section by considering one example of a $F^{(p+6)}$-form field strength background. The full potential is now

$$V = \frac{T_p}{\lambda^2}\int d^{p+1}x\Big[-\frac{1}{4}\sum_{a,b}\mathrm{Tr}\left([X^a,X^b]^2\right) + \\ + \frac{1}{5}F^{(p+6)}_{01\cdots pijklm}\,\mathrm{Tr}\left([X^i,X^j][X^k,X^l]X^m\right)\Big]\,. \tag{4.26}$$

Setting

$$F^{(p+6)}_{01\cdots pijklm} = -\frac{f}{\lambda}\epsilon_{ijklm}\,,\quad \{i,j,k,l,m\}\in\{p+1,p+2,p+3,p+4,p+5\} \tag{4.27}$$

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we have an extremum at

$$X^i = r\gamma^i\,, \quad i \in \{p+1, p+2, p+3, p+4, p+5\} \tag{4.28}$$

and,

$$r = \frac{5}{6}\frac{\lambda}{f}\,. \tag{4.29}$$

In (4.28) the γ_i's denote the Gamma matrices of SO(5) in an N-dimensional representation. One can verify that the extremum (4.28) is not a minimum, putting in an ansatz of the form (4.28) and varying r one finds a runaway direction as $r \to \infty$. It was argued in [31] that by taking the γ^i to be in the symmetric product of the four-dimensional representation one obtains a fuzzy generalization of S^4. This construction is quite different from the coset construction for fuzzy surfaces which is the main concern of this paper.

5. Supergravity duals

In this section we turn to constructing the supergravity descriptions of the fuzzy coset vacua. The gravity background for the $\mathcal{N} = 4$ theory is the near horizon geometry of an extremal D-brane. Deforming the $\mathcal{N} = 4$ theory corresponds to turning on additional perturbations in this background. No-hair theorem considerations suggest that such deformations give rise to singularities in general. In fact, we will find that singularities, corresponding to $(p+4)$- and $(p+6)$-dimensional brane sources, are present in the gravity duals. This makes the discussion below also of interest from the point of view of studying singularities in gravity via their gauge theory duals.

Our discussion follows [4] closely. These authors analyzed the gravity dual for the fuzzy S^2 case and showed that the gravity background contained 5-branes in the interior. This establishes that the infrared behavior of the configuration (3.6) and (3.11) is governed by the $(5+1)$-dimensional D5-brane theory (at least at large 't Hooft coupling). The strategy in [4] was to solve the gravity equations by choosing parameters which allowed spacetime to be divided into two region in which the stress-energy is dominated by the $(p+2)$-form field strength and $(p+4)$-form fields strengths, respectively. The first region is essentially a multi-centered version of the Dp-brane geometry while the second is the D$(p+2)$-brane metric. Denoting the two field strengths by $F^{(p+2)}$ and $F^{(p+4)}$, respectively, the first region corresponds to $(F^{(p+4)})^2/(F^{(p+2)})^2 \ll 1$ while the second to $(F^{(p+4)})^2/(F^{(p+2)})^2 \gg 1$. The crossover region between the two is described by the gravity background dual to the D$(p+2)$-brane with a U(1) (or equivalently NS B-field) turned on along its world volume,[12] [26, 27, 28]. This has a region of validity that overlaps with both the

[12] In the discussion below we will sometimes refer to this background as the non-commutative brane geometry.

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Dp- and D$(p+2)$-brane metrics. Here we will follows the same strategy. There will be one variation: the fuzzy surfaces correspond to Dp-branes distributed on higher-dimensional surfaces, accordingly the spacetime will sometimes be divided into more than two regions and branes of dimension $p+4$ and higher will enter the story as well. In the discussion below the required conditions on parameters will be found in a self-consistent manner. We assume a region of parameter space exists giving rise to some solution, construct the solution in parts, then deduce the required conditions on the parameters by demanding consistent overlap between the different parts.

One limitation of our analysis, mentioned in the introduction, needs to be pointed out here. In the solutions we construct, we show that the non-commutative $(p+2)$-brane metric and the p-brane metric overlap consistently only to leading order in the perturbation,[13] $(F^{(p+4)})^2/(F^{(p+2)})^2$. This is not enough to establish that the solutions exist. It is particularly important in the cases under discussion here to go further, because, as was mentioned in section 3, all supersymmetries are broken. Two arguments indicate that going to the next order in the perturbation should be enough to establish the existence or lack thereof of these solutions. In the gravity calculation, the radius of the surface is determined in terms of the perturbation $F^{(p+4)}$ only at the quadratic order. From the gauge theory perspective, since supersymmetry is broken, one expects that the most probable cause for destabilizing these surfaces is that the scalar fields acquire large masses — these are effects quadratic in the perturbation.[14] Unfortunately, the equations get rapidly complicated beyond leading order and we have not been able to push the analysis further. Accordingly the solutions we present here should be viewed as only a first step in a more complete analysis.[15]

We turn now to constructing the gravity backgrounds. For appropriate regions of parameter space, within the approximation mentioned above, we will find gravity solutions corresponding to $S^2 \times S^2$ and the two cosets of SU(3). The $S^2 \times S^2$ example is considered in some detail in the context of $AdS_5 \times S^5$. The cosets of SU(3), which require eight transverse dimensions, are discussed more briefly in the D1-brane background.

[13] Actually, as was mentioned above, higher-dimensional field strengths enter as well, but this is an inessential feature we suppress at the moment.

[14] In AdS/CFT correspondence, scalar masses are dual to Kaluza-Klein harmonics different from $F^{(p+4)}$, so if large mass terms are a concern, one might hope to stabilize the fuzzy surfaces by turning on appropriate values for these other modes as well. However, in the case of $AdS_5 \times S^5$, the traceless mass terms have supergravity duals but the trace component is dual to a string mode. Thus it is not clear that enough freedom available. For the D1-brane geometry considered below, the map between sugra modes and operators in the $\mathcal{N}=4$ theory is less well understood and we could not settle if all masses can be adjusted in the supergravity approximation.

[15] We should note that [4] does determine the radius of the two-sphere, which we mentioned above was sensitive to quadratic effects, but not from the gravity equations directly. Rather they consider a brane probe. Terms quadratic in the perturbation play an important role but their normalization can be determined by appealing to supersymmetry. In the present context, the absence of supersymmetry comes in the way of using this approach.

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5.1 Dual description of fuzzy $S^2 \times S^2$

The near horizon geometry of D3-branes is $AdS_5 \times S^5$. We are interested in the theory obtained by turning on additional trilinear terms (3.1) and (4.6), in the $\mathcal{N} = 4$ lagrangian. In the AdS/CFT correspondence a combination of this operator and the fermionic mass term is dual to the three-form field strength [21, 32, 33],

$$G_3 = F_3 - (C + ie^{-\phi})H_3 \,, \tag{5.1}$$

where F_3, H_3 stand for the RR and NS three-form field strengths, and C, ϕ for the axion and dilaton. Exactly, this case was studied in [4]. The main difference here is that the required perturbation (4.6) corresponds to turning on equal masses for the four gauginos,

$$m_{\lambda_i} = f_1 + if_2 \,, \tag{5.2}$$

and thus breaks supersymmetry completely.[16]

As mentioned above we choose parameters so that the spacetime can be divided into distinct regions each dominated by one p-form field strength. The effects of the additional perturbation die away close to the boundary and the geometry in this region is always the multi-centered D3-brane metric. We will see that deep in the interior, corresponding to the far infrared in the gauge theory, the geometry is a 7-brane one. This leaves room for two possibilities, both of which will be realized below. When one of the two spheres has a radius much smaller than the other, the D3-brane metric first goes over into a multi-centered version of the non-commutative 5-brane metric. In turn, proceeding further along the radial direction, this turns into the 7-brane metric. On the other hand when the two radii are more comparable, the D3-brane metric directly goes over to the 7-brane. We analyze these two cases in turn below.

Before proceeding, let us relate the parameters in the gravity solutions to those which appeared in section 4. (4.7) and (4.8) depend on four parameters, the strength of the perturbations f_1, f_2 and the size of the two SU(2) representations m, n. From our discussion in section 4.2, applied to the $p = 3$ case, it follows that (4.7) and (4.8), correspond to taking one 7-brane, and in this case $mn = N_3$ the number of 3-branes. Also, a U(1) field is turned on in the world volume of the 7-brane. The strength of this field is determined by m, n. On the gravity side we will find that the solutions depend on the two radii r_1, r_2 and on two parameters b_1, b_2 which specify the rank four NS B-field. The product, $b_1 b_2$ will be determined in terms of the number of 3- and 7-branes.

[16] In fact even in the limit when say $f_2 \to 0$ and the surface reduces to a single S^2 the background (4.6) does not preserve supersymmetry.

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5.1.1 D3 → D5 → D7

The multi-centered three-brane solution is

$$ds^2 = H_3^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H_3^{1/2}\sum_{m=1}^{6} dy^m dy_m\,, \qquad e^\phi = g_s\,,$$
$$F_5 = d\chi_4 + *d\chi_4\,, \quad \text{where} \quad \chi_4 = \frac{1}{gH_3}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,. \tag{5.3}$$

Here, H_3, the harmonic function is

$$H_3 = \frac{R_3^4}{16\pi^2}\int d\Omega_1 d\Omega_2 \frac{1}{|\vec r - \vec r(\Omega_1,\Omega_2)|^4}$$
$$\text{with} \qquad R_3^4 = 4\pi g_s N_3 l_s^4 \qquad \text{and} \qquad r^2 = y^m y_m\,. \tag{5.4}$$

H_3 corresponds to distributing the branes uniformly on the two S^2's. We take one sphere of radius r_1 to lie in the y_1, y_2, y_3 directions and the second sphere of radius r_2 to lie in the y_4, y_5, y_6 directions. Here, we also assume that, $r_2 \gg r_1$.

Besides the fields (5.3) the three form G_3 is also turned on. Asymptotically, as $r\to\infty$ this has the form:

$$G_3 = \alpha_3 r^{-4} + \beta_3 r^{-6} \tag{5.5}$$

with α_3 being the non-normalizable mode which is determined by the coefficient of the operator that is turned on in the gauge theory, and β_3 being the normalizable mode which corresponds to the five brane dipole moment. In addition, although we do not explicitly demonstrate it, the axion is also excited corresponding to the quadrupole seven brane moment discussed in section 4.2.

Now, let us approach the point $(y_4, y_5, y_6) = (0,0,r_2)$ close to the second sphere. Denote (y_1,y_2,y_3) by $\vec y$ and define $\rho^2 = \vec y^2 + (y_6 - r_2)^2$ to be the distance in the directions transverse to the two-sphere. For $r_1 \ll \rho \ll r_2$ the harmonic function takes the form:

$$H_3 \simeq \frac{1}{16\pi}\frac{R_3^4}{r_2^2}\int d\Omega_1 \frac{1}{[(w_3 - r_2)^2 + |\vec y - \vec y(\Omega_1)|^2]}\,, \tag{5.6}$$

where the integral is over the two-sphere of radius r_1.

Next consider the geometry for D5-branes extending along X_1, X_2, X_3, X_4, X_5 directions with an NS B-field turned on in the X_4, X_5 plane. We consider a multi-centered version of this geometry where the 5 branes are distributed in an S^2 of radius r_1 (lying in the y_1, y_2, y_3 directions)

$$ds^2 = H_5^{-1/2}\Big[-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + h(dx_4^2 + dx_5^2)\Big] + H_5^{1/2}(dy^m dy_m)\,,$$
$$e^\phi = g_s H_5^{-1/2}h^{1/2}\,, \qquad h^{-1} = 1 + \frac{b^2}{H_5 l_s^4}\,, \qquad B_{45} = \frac{b l_s^2}{l_s^4 H_5 + b^2}\,,$$
$$F^{(7)}_{012345r} = \frac{1}{g_s}\frac{l_s^2}{b}h\;\partial_r H_5^{-1}\,, \qquad F^{(5)}_{0123r} = \frac{1}{g_s}\partial_r H_5^{-1}\,, \tag{5.7}$$

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where the harmonic function H_5 is[17]

$$H_5(\vec{y}) = \frac{g_s N_5 b_2}{4\pi} \int d\Omega_2 \frac{1}{|\vec{y} - \vec{y}(\Omega_2)|^2} \,. \tag{5.8}$$

The parameter b_2 is related to the strength of the NS B-field and also determines $F^{(3)}$ and $F^{(5)}$.[18] It is easy to see that the effects of $F^{(5)}$ are negligible when $b_2^2/(l_s^4 H_5) \ll 1$ and dominate over $F^{(3)}, H_3$ when $b_2^2/(l_s^4 H_5) \gg 1$. To compare with the multi-centered D3-brane solution we therefore consider the region $b_2^2/(l_s^4 H_5) \gg 1$ in (5.7). In this region, $h \simeq H_5 l_s^4/b_2^2$ and the two metrics are the same. To see this identify y_1, y_2, y_3 in the two metrics; $(y_6 - r_2)$ in (5.3), with y_4 in (5.7). Finally, identify y_4, y_5 in (5.3), with x_4, x_5 in (5.7), after a rescaling. The two metric then agree, provided,

$$\frac{N_5}{\pi} \frac{b_2}{l_s^4} r_2^2 = N_3 \,. \tag{5.9}$$

Similarly the dilaton and $F^{(5)}$ also agree in this region. Verifying if the three-form field strength matches involves a subtlety. The three-form on the 3-brane side depends on an unknown parameter (which fixes the normalizable mode in the asymptotic region). To determined both this parameter and b_2 in terms of the non-normalizable mode's coefficient requires us to work to next to leading order in the perturbation $(G_3)^2/(F^{(5)})^2$. As mentioned at the beginning of this section, the second order calculation is beyond the scope of this paper. Before proceeding let us comment on (5.9). In the 5-brane theory the 3-brane density is determined by b_2/l_s^4. Since the 5-brane is wrapped on a sphere of radius r_2 (5.9) follows.

We now go to smaller values of $|\vec{y}|$, away from the crossover region in the 5-brane metric, (5.7). Once $b_2^2/(l_s^4 H_5) \ll 1$, $h \simeq 1$, and the solution reduces to the multi-centered 5-brane with no world volume B-field. Going to even smaller values of $|\vec{y}|$ one finds that the stress energy in the axion field begins to dominate, and (5.7) in turn crosses over to the 7-brane solution. The discussion of this transition is similar to the one above so we will be brief. The only new feature is that the the harmonic function in the 7-brane case varies logarithmically. The non-commutative 7-brane metric is given by

$$\begin{aligned} ds^2 &= H_7^{-1/2} \left(dx_0^2 + \sum_{i=1,\ldots,3} dx_i^2 + h_2(dx_4^2 + dx_5^2) + h_1(dx_6^2 + dx_7^2) \right) + \\ &\quad + H_7^{1/2}(d\rho^2 + \rho^2 d\phi^2) \\ \text{where} \quad H_7 &= \frac{C_7 g_s b_1 b_2}{l_s^4} N_7 \ \ln\left|\frac{2r_1}{\rho}\right| \qquad \text{and} \qquad h_1^{-1} = 1 + \frac{b_1^2}{H_7 l_s^4} \,. \end{aligned} \tag{5.10}$$

[17] From our discussion of dipole moments in section 4.2, it follows that in the gravity dual of (4.7) and (4.8), $N_5 = m$, more generally, $N_5 = N_7 m$.

[18] Once again, although we do not describe it explicitly, the axion is also excited in the solution (5.7).

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In addition, the axion and the three-form fields, H_3 and $F^{(3)}$ are also excited. C_7 above can be determined by ensuring that the axion has the correct periodicity in ϕ, for N_7 branes. After appropriately changing variable one can show that (5.10) agrees with the metric in (5.7) in the region $b_1^2 \gg H_7 l_s^4$, provided the condition

$$2C_7 N_7 \frac{b_1}{l_s^4} r_2^2 = N_5 \,, \tag{5.11}$$

is met. (5.11) is analogous to (5.9).

Let us summarize all the conditions required for solution described above to be valid. For the metric (5.3) and (5.7) to be both valid in the crossover region between the 5-brane and 3-brane solutions we have:

$$r_1 \ll \sqrt{\frac{N_5 g_s}{b_2}} l_s^2 \ll r_2 \,. \tag{5.12}$$

Substituting for b_2 from (5.9) yields

$$r_1 \ll \sqrt{\frac{g_s N_5^2}{\pi N_3}} r_2 \ll r_2 \,. \tag{5.13}$$

For (5.7) and (5.10) to have a common region of validity in turn implies (after dropping constants and taking the logarithm to be ~ 1):

$$\frac{b_1}{g_s N_7 b_2} \gg 1 \,. \tag{5.14}$$

Substituting for b_1, b_2 from (5.11) and (5.9) gives

$$\frac{1}{g_s} \frac{N_5^2}{N_7^2 N_3} r_2^2 \gg r_1^2 \,. \tag{5.15}$$

Finally, demanding that the curvature is small in string units except at the seven brane singularity, gives:

$$r_2 < \sqrt{N_3 g_s} r_1 \,. \tag{5.16}$$

(5.13), (5.15) and (5.16) are the independent constraints. A little thought shows that for $g_s \ll 1, N_3 \gg 1$, $N_7 \sim \mathrm{O}(1)$, they can all be met by appropriately choosing r_1, r_2 and N_5.[19]

5.1.2 D3 $\to$ D7

We now turn to considering the second possibility which is realized when the radii of the two S^2's are more comparable. Here, the D3-brane geometry directly goes over to the 7-brane solution. In this case let us consider the harmonic function (5.3)

[19] Or equivalently, from (5.9) r_1, r_2 and b_2.

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in the vicinity of the point $(y_1, y_2, y_3, y_4, y_5, y_6) = (0, 0, r_1, 0, 0, r_2)$. Define $\rho^2 = (y_3 - r_1)^2 + (y_6 - r_2)^2$, then for $\rho \ll r_1$ and $\rho \ll r_2$ we have

$$H_3 = \frac{\pi}{4} \frac{g_s N_3 l_s^4}{r_1^2 r_2^2} \ln\left[\frac{4 r_1^2 r_2^2}{(r_1^2 + r_2^2)\rho^2}\right]. \tag{5.17}$$

Compare this with the geometry for a 7-brane with a rank four NS-field turned on in its world volume. This solution is given by[20]

$$\begin{aligned} ds^2 &= \tilde{H}_7^{-1/2}\left[-dx_0^2 + \sum_{i=1,\dots,3} dx_i^2 + h_1(dx_4^2 + dx_5^2) + h_2(dx_6^2 + dx_7^2)\right] + \\ &\quad + \tilde{H}_7^{1/2}(d\rho^2 + \rho^2 d\phi^2)\,, \end{aligned} \tag{5.18}$$

$$\text{where} \quad \tilde{H}_7 = \frac{C_7 g_s b_1 b_2}{l_s^4} N_7 \ln\left[\frac{4 r_1^2 r_2^2}{(r_1^2 + r_2^2)\rho^2}\right], \qquad \text{and} \qquad h_i^{-1} = 1 + \frac{b_i^2}{H_7 l_s^4}, \quad i = 1, 2\,.$$

In the region where

$$b_i^2 \gg H_7 l_s^4 \tag{5.19}$$

the two metrics (5.18), (5.3) and (5.17), (as well as other fields like the dilaton, $F^{(5)}$ etc. which we do not explicit exhibit) agree provided,

$$\frac{4C_7}{\pi} \frac{N_7 r_1^2 r_2^2 b_1 b_2}{l_s^8} = N_3\,. \tag{5.20}$$

For (5.19) to be true we have, approximating the logarithm by unity and dropping various constants, that

$$N_7 g_s \ll \frac{b_1}{b_2} \qquad \text{and} \qquad N_7 g_s \ll \frac{b_2}{b_1}\,. \tag{5.21}$$

Finally, the requirement that the curvature is small until one comes close to the seven brane is met once $g_s N_3 \gg 1$. In summary for the solution (5.18) and (5.3), to exist, r_1, r_2, b_1, b_2 must thus satisfy the conditions (5.21) and (5.20).

Let us end with two comments. First, one can use S-duality to generate additional solutions, both for the case in section (5.12) and here. These are valid in different (and somewhat complimentary) regions of parameter space. Second, as was mentioned at the outset of this section, our analysis in the various crossover regions has been to linear order in the perturbing RR potential. We need to go beyond this, at least to the next order, before conclusively establishing the existence, or lack thereof, of these solutions. Since supersymmetry is broken, it is quite likely, that such a second order analysis will reveal that some of the solutions constructed here are unstable. But hopefully, some will survive, yielding gravity backgrounds duals to fuzzy surfaces.

[20] Once again the coefficient C_7 can be fixed by demanding periodicity in ϕ.

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5.2 Gravity duals to cosets of SU(3)

We turn next to the cosets of SU(3). In this case one needs at least eight transverse dimensions. We will work with the D1-brane system below. Holography is not as well understood in this context as it is for the D3-brane system but this is not a big limitation for our limited analysis below. Our discussion will be somewhat brief, since many of the essential points have been covered above.

We start with the $\mathbb{CP}^2$ case then turn to $\mathrm{SU}(3)/(\mathrm{U}(1)\times\mathrm{U}(1))$ in the next section. In section 4.3 we saw that the irreducible representation was completely determined by the dimension N, with $m^2 = 2N$. The discussion on the dipole moments (adapted for the D1-brane system here) also showed that the irreducible representation corresponds to taking the number of 5-branes, which wrap $\mathbb{CP}^2$, $N_5 = 1$. To clarify the origin of various terms, we keep N_5 as a free parameter below. Also, here we will restrict ourselves to the simplest case where spacetime gets divided into only two regions, one being the D1-brane metric and the other the D5-brane geometry.

5.2.1 $\mathbb{CP}^2$

The multi-centered near-horizon limit of the D1-brane geometry is given by:

$$ds^2 = H_1^{-1/2}(-dx_0^2 + dx_1^2) + H^{1/2}\left(\sum (dX^i)^2\right), \qquad e^{2\phi} = g_s^2 H_1\,, \tag{5.22}$$

where X_i denote the eight transverse coordinates. H_1 is the harmonic function which corresponds to uniformly distributing the D1-branes over a transverse $\mathbb{CP}^2$ surface. From appendix B we have

$$H_1(\vec{r}) = \frac{1}{2\pi^2} R_1^6 \int \frac{d^2 z_1 d^2 z_2}{(1+|z_1|^2+|z_2|^2)^3} \frac{1}{|\vec{r} - \vec{r}(z_1,z_2)|^6}\,, \qquad R_1^6 = 32\pi^2 g_s N_1 l_s^6\,. \tag{5.23}$$

$\vec{r}$ is the eight-dimensional transverse vector, and $\vec{r}(z_1,z_2)$ is determined by (B.14) and (B.15).

We are interested in the deformed $\mathcal{N} = 4$ theory, this is dual to a background with a $F^{(5)}$ field strength perturbation turned on. In addition as shown in section 4 a dipole moment for the $F^{(7)}$ field strength is also expected, this is the magnetic dual to $F^{(3)}$, so the $F^{(3)}$ field should also change from its value in the $\mathcal{N} = 4$ case. We do not exhibit these perturbations explicitly below.

We now approach the point $z_1, z_2 = 0$ on $\mathbb{CP}^2$. At this point, we see from (B.14), that only X_3, X_8 are non zero. Let us denote their values to be X_3^0, X_8^0, respectively. From (B.14) we also see that at this point the coordinates parallel to the surfaces are X_1, X_2, X_3, X_4. Define $\rho^2 = (X_8 - X_8^0)^2 + (X_3 - X_3^0)^2 + X_5^2 + X_6^2$ to be the normal distance. Then, in the vicinity of this point,

$$H_1 = \frac{16}{81}\frac{R_1^6}{R^4}\frac{1}{\rho^2}\,, \tag{5.24}$$

where R is the radius of the surface, (4.15).

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In comparison the D5-brane metric with a rank four B-field turned on is

$$ds^2 = H_5^{-1/2}\left[-dx_0^2 + dx_1^2 + h_1(dx_2^2 + dx_3^2) + h_2(dx_4^2 + dx_5^2)\right] + H_5^{1/2}\left[\sum_{m=6}^{9} dy_m dy^m\right],$$

$$h_i^{-1} = 1 + \frac{b_i^2}{H_5 l_s^4}, \quad H_5 = \frac{R_5^2}{r^2}, \quad R_5^2 = \frac{g_s b_1 b_2 N_5}{l_s^2}. \tag{5.25}$$

Our discussion of $F^{(3)}$ dipole moments, section 4.3, implies that

$$b_1 = b_2 \equiv b\,. \tag{5.26}$$

Now in the region where $b_i^2 \gg H_5 l_s^4$, one can show, after a suitable change of variables, that (5.22) and (5.25) agree provided,

$$\frac{81}{512\pi^2}\frac{N_5 b^2 R^4}{l_s^8} = N_1\,. \tag{5.27}$$

(5.27) is analogous to (5.9).

The other conditions which need to be met for this solution to be valid are as follows. An overlapping region of validity for (5.25) and (5.22), requires:

$$R^2 \gg g_s N_5 l_s^2\,. \tag{5.28}$$

For the curvature to be small in string units except close to the 5-brane and for the string coupling to be small in the overlap region requires:

$$g_s N_1^{1/3} \ll \frac{R^2}{l_s^2} \ll g_s N_1\,. \tag{5.29}$$

For $N_5 = 1$, $g_s \ll 1$ and $N_1 \gg 1$ these can all be met.

5.2.2 SU(3)/(U(1) × U(1))

Finally we discuss briefly the SU(3)/(U(1) × U(1)) case. Here, the harmonic function (5.23) is replaced by the one appropriate to distributing the branes on this coset. This can be determined by (C.5), in appendix C. Approaching the point $z_i = 0$ we find that this metric crosses over to a seven-brane metric of the form:

$$\begin{aligned} ds^2 &= H_7^{-1/2}\Big[-dx_0^2 + dx_1^2 + h_1(dx_2^2 + dx_3^2) + h_2(dx_4^2 + dx_5^2) + h_3(dx_6^2 + dx_7^2)\Big] + \\ &\quad + H_7^{1/2}(d\rho^2 + \rho^2 d\phi^2)\,, \end{aligned} \tag{5.30}$$

$$h_i^{-1} = 1 + \frac{b_i^2}{H_7 l_s^4}, \quad H_7 = \tilde{C}_1 \frac{g_s b_1 b_2 b_3}{l_s^6} N_7 \ln\left[\frac{\tilde{C}_2 R^2}{\rho^2}\right],$$

$$\text{provided} \quad \frac{N_7 b_1 b_2 b_3 R^6}{l_s^{12}} \sim N_1\,. \tag{5.31}$$

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Here R is the radius of the surface, defined in (4.17). $\tilde{C}_{1,2}$ in (5.30) are coefficients which can be determined as in (5.18). The metric (5.30) depends on the parameters b_i's. The product is determined from (5.31) in terms of R^2. The ratios b_i/b_j , we saw in our discussion in section 4.3 are determined in terms of m, n — the two integers specifying the representation of SU(3). There are additional conditions, for an overlapping region of validity for (5.22) and (5.30), for the the string coupling to be small and for the curvature to be small except at the brane singularity. When $m/n \sim \mathrm{O}(1)$, these can all be met for $N_7 = 1$, $g_s \ll 1$, $N_1 \gg 1$.

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A. SU(3) and its representations

We follow the conventions of [34]. The standard basis for the defining (or fundamental) representation of SU(3) consists of the Gell-Mann matrices λ_a, $a = 1, \ldots 8$, the generators being $\lambda_a/2$. The *complex conjugate* (or anti-fundamental) representation has generators $-\lambda_a^*/2$. We will denote the gererators as T^a, and it will be clear from the context as to which representation we are referring to.

The group SU(3) has 2 simple roots, denoted by

$$\alpha^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad \alpha^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \tag{A.1}$$

where the first entry is the eigenvalue of T_3 and the second entry the eigenvalue of T_8 in the adjoint representation. The 3 roots of SU(3) are α^1, α^2 and $\alpha^3 = \alpha^1 + \alpha^2$, and the corresponding root vectors are denoted by E_{α^i} below.

The fundamental weights (with the above choice of α_i's) are

$$\mu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \qquad \mu^2 = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right). \tag{A.2}$$

The highest weight of the defining representation is μ^1 while that of the anti-fundamental representation is μ^2. The lowest weight vector of the fundamental representation is $-\mu^2$ and of the anti-fundamental representation is $-\mu^1$.

The unitary irreducible representations (UIR's) of SU(3) are labelled by a pair of integers (n, m). The states in the UIR (n, m) are constructed by taking the tensor product of n fundamental representations (i.e. $(1, 0)$) and m anti-fundamental

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representations (i.e. $(0,1)$). The lowest weight state $|\mathrm{lws}\rangle_{(n,m)}$ for (n,m) is

$$|\mathrm{lws}\rangle_{(n,m)} = \underbrace{|\mathrm{lws}\rangle_{(1,0)} \otimes \cdots |\mathrm{lws}\rangle_{(1,0)}}_{n \text{ factors}} \underbrace{|\mathrm{lws}\rangle_{(0,1)} \otimes \cdots |\mathrm{lws}\rangle_{(0,1)}}_{m \text{ factors}} . \tag{A.3}$$

All other states of (n,m) can be obtained by by applying the raising operators repeatedly.

B. SU(3) coherent states

We closely follow [15] for notation and conventions in the construction of coherent states.

For any state $|\mu\rangle$ corresponding to a weight vector μ, the isotropy subgroup H_μ contains the Cartan subgroup $H = \mathrm{U}(1) \times \mathrm{U}(1)$. For general weight vectors, H_μ coincides with H, and the coherent state is characterized by the point of $\mathcal{M} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$. However, SU(3) also has *degenerate* representations for which the lowest weight μ is singular, i.e. $\alpha \cdot \mu = 0$ for some root α. Consider for example $|\mathrm{lws}\rangle_{(0,m)}$. In this case, the isotropy group is not $\mathrm{U}(1) \times \mathrm{U}(1)$ but $\mathrm{U}(2)$, and the coherent state is characterized by a point of $\frac{\mathrm{SU}(3)}{\mathrm{U}(2)} = \mathbb{CP}^2$.

To construct a coherent state, we start with the vector $|\mathrm{lws}\rangle_{(n,m)}$. This state satisfies $E_{-\alpha^i}|\mathrm{lws}\rangle_{(n,m)} = 0$, where α^i's are roots and $E_{-\alpha^i}$ are the lowering operators. SU(3) has 3 roots: α^1, α^2 and $\alpha^3 = \alpha^1 + \alpha^2$. For both degenerate as well as non-degenerate representations, the coherent state is defined as

$$|x\rangle = T(g)|\mathrm{lws}\rangle \,, \tag{B.1}$$

where $T(g)$ is the representation of the element $g \in \mathrm{SU}(3)$. Equivalently, It can also be written as

$$|\xi\rangle = N(\xi,\bar{\xi}) \exp\left[\sum_i \xi_i E_{\alpha^i}\right] |\mathrm{lws}\rangle_{(n,m)} \,, \tag{B.2}$$

where E_{α^i} are the raising operators. The normalization N is determined (up to an overall phase) by the condition

$$\langle\xi|\xi\rangle = 1 \,. \tag{B.3}$$

B.1 Coherent states corresponding $\mathbb{CP}^2$

We start with the $(0,m)$ representation of SU(3). The raising/lowering operators can be defined as

$$\frac{T_1 \pm iT_2}{\sqrt{2}} \equiv E_{(\pm1,0)} \equiv T_\pm \,, \tag{B.4}$$

$$\frac{T_4 \pm iT_5}{\sqrt{2}} \equiv E_{(\pm1/2,\pm\sqrt{3}/2)} \equiv U_\pm \,, \tag{B.5}$$

$$\frac{T_6 \pm iT_7}{\sqrt{2}} \equiv E_{(\mp1/2,\pm\sqrt{3}/2)} \equiv V_\pm \,. \tag{B.6}$$

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Using the lowest weight state $|\mathrm{lws}\rangle$ we can construct coherent states

$$|\xi_1, \xi_2\rangle = N e^{\xi_1 T_+ + \xi_2 U_+} |\mathrm{lws}\rangle \equiv |\xi_i\rangle\,, \tag{B.7}$$

where $T_\pm$ are the usual raising/lowering operators. The normalization N can be determined (up to an overall phase) to be

$$|N| = \left[1 + \frac{|\xi_1|^2 + |\xi_2|^2}{2}\right]^{-m/2}\,. \tag{B.8}$$

Using the one-form $\mathcal{A}$ defined as

$$\mathcal{A} = \langle \chi_1, \chi_2 | d | \xi_1, \xi_2\rangle|_{\chi_i = \xi_i}\,, \quad \text{where} \quad d|\xi_1,\xi_2\rangle = \partial_{\xi_i}|\xi_1,\xi_2\rangle d\xi_i + \partial_{\bar{\xi}_i}|\xi_1,\xi_2\rangle d\bar{\xi}_i\,. \tag{B.9}$$

we can calculate the Kähler form

$$K = d\mathcal{A}\,. \tag{B.10}$$

To simplify equations, we define $z_i = \xi_i/\sqrt{2}$ and work with z_i's henceforth. Then we find that

$$\begin{aligned} \mathcal{A} &= d\ln N + \langle \xi_i | T_+ | \xi_i\rangle d\xi_1 + \langle \xi_i | U_+ | \xi_i\rangle d\xi_2\,, && \text{(B.11)} \\ &= d\ln N + \frac{m\bar{z}_1}{1 + z_i\bar{z}_i} dz_1 + \frac{m\bar{z}_2}{1+z_i\bar{z}_i} dz_2\,. && \text{(B.12)} \end{aligned}$$

Then the Kähler form K is

$$K = d\mathcal{A} = \frac{m}{(1+z_k\bar{z}_k)^2}\Big[(1+z_k\bar{z}_k)\delta_{ij} - z_i\bar{z}_j\Big] d\bar{z}_i \wedge dz_j\,. \tag{B.13}$$

We will also need the following:

$$\begin{aligned} \langle z_i|T_+|z_i\rangle &= \left(\frac{m}{2}\right)\frac{\sqrt{2}\bar{z}_1}{1+z_i\bar{z}_i}\,, & \langle z_i|T_-|z_i\rangle &= \left(\frac{m}{2}\right)\frac{\sqrt{2}z_1}{1+z_i\bar{z}_i}\,, \\ \langle z_i|U_+|z_i\rangle &= \left(\frac{m}{2}\right)\frac{\sqrt{2}\bar{z}_2}{1+z_i\bar{z}_i}\,, & \langle z_i|U_-|z_i\rangle &= \left(\frac{m}{2}\right)\frac{\sqrt{2}z_2}{1+z_i\bar{z}_i}\,, \\ \langle z_i|V_+|z_i\rangle &= \left(\frac{m}{2}\right)\frac{\sqrt{2}\bar{z}_1 z_2}{1+z_i\bar{z}_i}\,, & \langle z_i|V_-|z_i\rangle &= \left(\frac{m}{2}\right)\frac{\sqrt{2}z_1\bar{z}_2}{1+z_i\bar{z}_i}\,, \\ \langle z_i|T_3|z_i\rangle &= \left(\frac{m}{2}\right)\frac{(|z_1|^2-1)}{1+z_i\bar{z}_i}\,, & \langle z_i|T_8|z_i\rangle &= \frac{(m/2)}{\sqrt{3}}\frac{(2|z_2|^2-|z_1|^2-1)}{1+z_i\bar{z}_i}\,. \end{aligned} \tag{B.14}$$

Let us end with one comment. (4.2), shows that the coordinates X^{p+i} are proportional to the generators T^i. From (B.14) we then learn that $\mathbb{CP}^2$ is embedded in $\mathbb{R}^8$ as follows:

$$X^{p+a}(z_i) = f\langle z_i|T^a|z_i\rangle\,, \quad a = 1,\ldots,8\,. \tag{B.15}$$

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C. Coherent states for $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$

Again, start with the lowest weight state $|\mathrm{lws}\rangle \equiv |0\rangle$ of an arbitrary UIR (n, m) of SU(3). The coherent state is defined as

$$|\xi\rangle = Ne^{\sum \xi_i E_{\alpha^i}}|\mathrm{lws}\rangle \,. \tag{C.1}$$

It is easy to show that

$$\exp \xi_i E_{\alpha^i} = \exp \xi_1 E_{\alpha^1} \cdot \exp \left(\xi_2 + \frac{\xi_1\xi_3}{2\sqrt{2}}\right) E_{\alpha^2} \cdot \exp \xi_3 E_{\alpha^3} \,. \tag{C.2}$$

To minimize irritating factors of $\sqrt{2}$, we define $z_1 = \xi_1/\sqrt{2}$, $z_2 = \xi_2/\sqrt{2} + \xi_1\xi_3/4$ and $z_3 = \xi_3/\sqrt{2}$.

We grind through to find the following quantities:

$$|N|^{-2} = \left[\frac{1 + |z_1|^2 + |z_2 - z_1z_3|^2}{1 + |z_2|^2 + |z_3|^3}\right]^{n+m} \cdot [1 + |z_2|^2 + |z_3|^2]^m \,. \tag{C.3}$$

For notational simplicity, we define

$$A(z_i, \bar{z}_i) = 1 + |z_1|^2 + |z_2 - z_1z_3|^2 \,, \qquad B(z_i, \bar{z}_i) = 1 + |z_2|^2 + |z_3|^2 \,. \tag{C.4}$$

We also need the following expectation values:

$$\begin{aligned}
\langle z_i|T_1|z_i\rangle &= \frac{(m+n)}{2}\left[\frac{\bar{z}_1 + z_1}{A(z_i, \bar{z}_i)}\right] - \frac{n}{2}\left[\frac{z_3\bar{z}_2 + z_2\bar{z}_3}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_2|z_i\rangle &= \frac{(m+n)}{2i}\left[\frac{\bar{z}_1 - z_1}{A(z_i, \bar{z}_i)}\right] - \frac{n}{2i}\left[\frac{z_3\bar{z}_2 - z_2\bar{z}_3}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_3|z_i\rangle &= \frac{(m+n)}{2}\left[\frac{|z_1|^2 - 1}{A(z_i, \bar{z}_i)}\right] - \frac{n}{2}\left[\frac{|z_2|^2 - |z_3|^3}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_4|z_i\rangle &= \frac{(m+n)}{2}\left[\frac{(z_2 - z_1z_3) + (\bar{z}_2 - \bar{z}_1\bar{z}_3)}{A(z_i, \bar{z}_i)}\right] - \frac{n}{2}\left[\frac{z_2 + \bar{z}_2}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_5|z_i\rangle &= \frac{(m+n)}{2i}\left[\frac{(\bar{z}_2 - \bar{z}_1\bar{z}_3) - (z_2 - z_1z_3)}{A(z_i, \bar{z}_i)}\right] - \frac{n}{2i}\left[\frac{\bar{z}_2 - z_2}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_6|z_i\rangle &= \frac{(m+n)}{2}\left[\frac{z_1\bar{z}_2 + z_2\bar{z}_1 + |z_1|^2(\bar{z}_3 + z_3)}{A(z_i, \bar{z}_i)}\right] + \frac{n}{2}\left[\frac{\bar{z}_3 + z_3}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_7|z_i\rangle &= \frac{(m+n)}{2i}\left[\frac{z_2\bar{z}_1 - z_1\bar{z}_2 + |z_1|^2(\bar{z}_3 - z_3)}{A(z_i, \bar{z}_i)}\right] + \frac{n}{2i}\left[\frac{\bar{z}_3 - z_3}{B(z_i, \bar{z}_i)}\right], \\
\langle z_i|T_8|z_i\rangle &= \frac{(m+n)}{2\sqrt{3}}\left[\frac{2|z_2 - z_2z_3|^2 - (|z_1|^2 - 1)}{A(z_i, \bar{z}_i)}\right] + \frac{n}{2\sqrt{3}}\left[\frac{|z_2|^2 + |z_3|^3 - 2}{B(z_i, \bar{z}_i)}\right].
\end{aligned} \tag{C.5}$$

To calculate the Kähler form, we also need

$$|N|^2\langle 0|e^{\bar{\xi}_i E_{-\alpha^i}} e^{\xi_i E_{\alpha^i}} E_{\alpha^3}|0\rangle = -n\frac{\bar{z}_3 + z_1\bar{z}_2}{B(z_i, \bar{z}_i)} \,. \tag{C.6}$$

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Putting it all together to calculate the Kähler form, we get

$$\begin{aligned}K = &\frac{m+n}{A}(d\bar{z}_1 \wedge dz_1 + d\bar{z}_2 \wedge dz_2) - \frac{m+n}{A^2} dA \wedge (\bar{z}_1 dz_1 + (\bar{z}_2 - \bar{z}_1\bar{z}_3)dz_2) + \\ &+ \frac{n}{B^2} dB \wedge (z_3\bar{z}_2 dz_1 + \bar{z}_2 dz_2) - \frac{n}{B}(z_3 d\bar{z}_2 \wedge dz_1 + \bar{z}_2 dz_3 \wedge dz_1 + d\bar{z}_2 \wedge dz_2) - \\ &- \frac{n}{B}(d\bar{z}_3 \wedge dz_3 + z_1 d\bar{z}_2 \wedge dz_3 + \bar{z}_2 dz_1 \wedge dz_3) + \frac{n}{B^2}(\bar{z}_3 + z_1\bar{z}_2) dB \wedge dz_3\,. \end{aligned} \tag{C.7}$$

It is easy to see that in the limit $m \to \infty, n/m \to 0$, (and doing a coordinate redefinition $(z_2 - z_1 z_3) \to z_2$) (C.4) go over into (B.14), and (C.7) goes over to into (B.13). Thus in this limit, $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ degenerates to $\mathbb{CP}^2$.

Finally, as in the $\mathbb{CP}^2$ case above we can express X^{p+a} (4.2), as a function of the coordinate z_i as

$$X^{p+a} = \langle z_i | T^a | z_i \rangle\,. \tag{C.8}$$

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